Path-Dependent Mid-Term Debt Defaults

Gary Schurman MBE, CFA

July 2021

Most option-based default models assume that the default occurs at debt maturity when debt principal and interest are due and payable to the lender. The borrower is assumed to default end-of-term if asset value (debt collateral) is less than debt value at that time (i.e. the borrower is insolvent). What is ignored are scenarios where prior to debt maturity asset value crosses some default barrier where the borrower determines that it is in the borrower's best interest to default mid-term rather then wait until end-of-term to make that decision.

We will define the time variable T to be debt maturity in years. The default barrier is an amount less than debt value (borrower is technically insolvent) where the borrower determines that rather than hope that asset value goes above debt value at time T the borrower will default at time $t < T$ and save the debt service payments scheduled to be made over the time interval $[t, T]$.

We will define mid-term defaults to be scenarios where asset value goes below some default barrier value over the time interval $[0, T]$ and ends up above debt value at time T. These values would be added to the results of option-based models that consider end-of-term defaults only. In this white paper we will build a model that considers path-dependent mid-term defaults.

Our Hypothetical Problem

We are tasked with building a model to calculate the effect of mid-term defaults as defined above. We are given the following go-forward model assumptions...

Table 1: Go-Forward Model Assumptions

Symbol	Description	Value
A_0	Asset Value at time zero ($)	6,000,000
D_0	Debt Value at time zero ($)	4,500,000
μ	Annual expected return - mean (%)	8.00
σ	Annual expected return - volatility (%)	20.00
ϕ	Annual expected return - dividend yield (%)	2.50
Δ	Barrier default factor (% of debt value)	85.00
ω	Collateral recovery rate (%)	90.00
T	Debt maturity in years (#)	5.00

Our task is to answer the following questions given that the risk-free rate is 3.00%...

Question 1: What is the risk-neutral probability of a mid-term default?

Question 2: What is the loss given default?

Question 3: What is the amount that should be added to the results of option-based models that consider end-of-term defaults only?

Asset Value and Debt Value

We will define the variables θ, α and υ to be random asset return, expected return mean, and expected return variance, respectively, over the time interval $[0, T]$. The equations for these return variables are...

$$\text{if... } \alpha = \left(\mu - \phi - \frac{1}{2}\sigma^2\right)T \text{ ...and... } \upsilon = \sigma^2 T \text{ ...then... } \theta = \alpha + \sqrt{\upsilon}\, z \text{ ...where... } z \sim N\Big[0, 1\Big] \tag{1}$$

We will define the variable A_T to be asset value at time T, which is a function of asset value at time zero and random asset return over the time interval $[0, T]$. Using Equation (1) above the equation for random asset value at time T is...

$$A_T = A_0 \, \mathrm{Exp}\left\{\theta\right\} \text{ ...where... } \theta \sim N\left[\alpha, \upsilon\right] \tag{2}$$

Given that random asset return is normally-distributed, random asset value is lognormally-distributed. Using Equations (1) and (2) above the equation for expected asset value at time T under the actual probability Measure P is...

$$\mathbb{E}^P\left[A_T\right] = A_0 \, \mathrm{Exp}\left\{\alpha + \frac{1}{2}\,\upsilon\right\} = A_0 \, \mathrm{Exp}\left\{(\mu - \phi)\,T\right\} \tag{3}$$

We will define the non-random variable D_T to be debt value at time T. The equation for debt value at time T is...

$$D_0 = D_T = \text{A constant value} \tag{4}$$

Probability of Default

We will define default point x to be the value of the random return where asset value at time T equals debt value at time T. Using Equations (2) and (4) above the equation for default point x is...

$$\text{if... } A_0 \, \mathrm{Exp}\left\{x\right\} = D_T \text{ ...then... } x = \ln\left(\frac{D_T}{A_0}\right) \tag{5}$$

We will define default point y to be the value of the random return where asset value equals debt value times the barrier default factor. Using Equations (2) and (4) above and the model parameters in Table 1 above the equation for default point y is...

$$\text{if... } A_0 \, \mathrm{Exp}\left\{y\right\} = \Delta\, D_T \text{ ...then... } y = \ln\left(\frac{\Delta\, D_T}{A_0}\right) \text{ ...where... } 0 < \Delta < 1 \tag{6}$$

We will define the variable m to be the minimum value of the random return over the time interval $[0, T]$ and the variable w to be the random return at time T. We will define the function $a(m, w)$ to be the joint distribution function of m and w. Using Equation (1) above the equation for the joint distribution function is... [2]

$$a(m, w) = \frac{2\,(w - 2m)}{\upsilon\sqrt{2\pi\upsilon}} \, \mathrm{Exp}\left\{\frac{\alpha w}{\upsilon} - \frac{\alpha^2}{2\upsilon} - \frac{1}{2\upsilon}(2m - w)^2\right\} \tag{7}$$

We will define default point x to be the threshold value and default point y to be the default barrier value. Using Equations (5), (6) and (7) above the equation for the probability that the **minimum value** of random asset return goes below the default barrier value **y** over the time interval $[0, T]$ and the **ending value** of random asset return is above the threshold value **x** at time T is... [3]

$$\mathrm{Prob}\left[\mathrm{Min}(\theta) \leq y, \theta \geq x\right] = \int_{w=x}^{w=\infty} \int_{m=-\infty}^{m=y} a(m, w)\, \delta m \, \delta w \tag{8}$$

We will define the function CNDF(x, mean, variance) to be the cumulative normal distribution function of the normally-distributed random variable x. The Excel equivalent to the cumulative normal distribution function is...

$$CNDF(x, mean, variance) = NORMDIST(x, mean, \sqrt{variance}, true) \text{ ...where... } x \sim N\left[mean, variance\right] \tag{9}$$

Using Equations (1) and (9) above the solution to Equation (8) above is...

$$\text{Prob Default} = \mathrm{Exp}\left\{\frac{2\,\alpha\, y}{\upsilon}\right\}\left(1 - CNDF\left[x, \alpha + 2\,y, \upsilon\right]\right) \tag{10}$$

Loss Given Default

If asset value goes below the default barrier y as defined by Equation (6) above then the borrower will default mid-term. Using Equations (2) and (6) above the equation for loss given default at time T is...

$$\text{if... } \theta_T = y \text{ ...then... } A_T = A_0 \,\mathrm{Exp}\left\{ \ln\left(\frac{\Delta\, D_T}{A_0}\right)\right\} = \Delta\, D_T \tag{11}$$

Using Equation (11) above and the parameters in Table 1 above the equation for loss given default at time T is...

$$\mathrm{LGD} = D_T - \omega\, A_T = D_T\Bigg(1 - \omega\,\Delta\Bigg) \tag{12}$$

We will define the variable κ to be the discount rate. Using Equation (12) above the equation for the present value of the loss given default is...

$$\mathrm{PV\ LGD} = D_T\Bigg(1 - \omega\,\Delta\Bigg)\mathrm{Exp}\Bigg\{ -\kappa\, T\Bigg\} \tag{13}$$

The Answers To Our Hypothetical Problem

Using Equation (1) above return mean and variance are...

$$\alpha = \Bigg(0.0300 - 0.0250 - \frac{1}{2} \times 0.2000^2\Bigg) \times 5 = -0.07500 \text{ ...and... } \upsilon = 0.2000^2 \times 5 = 0.20000 \tag{14}$$

Note that the variable μ under the risk-neutral probability Measure Q is the risk-free rate.

Using Equations (5) and (6) above and the parameters in Table 1 above the equations for default points x and y are...

$$x = \ln\left(\frac{4,500,000}{6,000,000}\right) = -0.28768 \text{ ...and... } y = \ln\left(\frac{0.85 \times 4,500,000}{6,000,000}\right) = -0.45020 \tag{15}$$

Question 1: What is the risk-neutral probability of a mid-term default?

Using Equations (10), (14) and (15) above the answer to the question is...

$$\mathrm{Prob} = \mathrm{Exp}\left\{\frac{2 \times -0.075 \times -0.4502}{0.2000}\right\} \times \Bigg(1 - CNDF\Bigg[-0.28768, -0.075 + 2 \times -0.4502, 0.2000\Bigg]\Bigg) = 0.08697 \tag{16}$$

Question 2: What is the loss given default?

Using Equation (12) above the equation for the answer to the question is...

$$\mathrm{LGD} = 4,500,000 \times \Bigg(1 - 0.90 \times 0.85\Bigg) = 1,057,500 \tag{17}$$

Question 3: What is the amount that should be added to the results of option-based models that consider end-of-term defaults only?

Using Equations (13) and (16) above the answer to the question is...

$$\text{PV of Expected Loss} = 0.08697 \times 1,057,500 \times \mathrm{Exp}\Bigg\{ -0.0300 \times 5\Bigg\} = 79,162 \tag{18}$$

References

[1] Integral Solutions - Integral One, Schurman, July 2021

[2] Integral Solutions - Integral Two, Schurman, July 2021

[3] Integral Solutions - Integral Three, Schurman, July 2021